Explanation File of Lagrange Program

The iterative methods (Lagrange or Newton) to find a real root of a non-linear function f(x) can be put in the general form:

$$\begin{cases} \text{For a given } x_0, \\ x_{i+1} = g(x_i) \end{cases}$$

We must choose function g(x) so that the sequence x0, x1, x2,…xi, x i+1… tends to the root xa when I increases indefinitely: xi must converge to xa, root of equation f(x) = 0.
This means that the solution of g(x) = x in interval [a,b] must be the same than that of f(x)=0 in the same interval.

A function g is said "contractive" if it exists a constant L, such as for any x and y values taken in interval [a,b]:

$$|g(x) - g(y)| <= L * |x-y| \quad \text{with } 0 <= L < 1$$

If function g has a derivative, it is sufficient, for g to be contractive, that |g'(x)| < 1 for any x value taken in interval [a,b].
Finally, we can demonstrate (not done here) that, still in interval [a,b], if g is continuous, contractive then, for any x0 in interval [a,b], the sequence xi converges towards the unique solution xa of x = g(x) with xa in [a,b].

In the case of the Lagrange method, we take for g(x):

$$g(x) = \frac{A*f(x) - x*f(a)}{f(x) - f(a)}$$

If f has a first derivative :

$$g'(x) = \frac{[a*f'(x)-f(a)]*[f(x)-f(a)] - f'(x)*[a*f(x)-x*f(a)]}{[f(x) - f(a)]^2}$$

Since f(xa)=0, xa is a root of f(x), we have:

$$g'(xa) = \frac{f(a) + (xa-a)f'(xa)}{f(a)}$$

This can be put under the form (*):

$$g'(xa) = \frac{(a-xa)^2 * f''(c)}{2 * f(a)} \qquad a <= c <= xa$$

The convergence is obtained when g'(xa) < 1, we must also have the conditions:

$$f''(c) * f(a) > 0 \text{ and } f(a) <> 0.$$

(*) By using the Taylor formula:

$$f(a) = f(xa) + (a-xa)*f'(xa) + ((a-xa)^2/2)*f''(c) \quad \text{with } a <= c < xa.$$

[From BIBLI07].